

# ON GORENSTEIN GLOBAL DIMENSION IN TRIVIAL RING EXTENSIONS

NAJIB MAHDOU AND MOHAMMED TAMEKKANTE

ABSTRACT. In this paper, we compare the Gorenstein homological dimension of a ring  $R$  and of its trivial ring extension by an module  $E$ .

## 1. INTRODUCTION

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Let  $R$  be a ring, and let  $M$  be an  $R$ -module. As usual we use  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$  and  $\text{fd}_R(M)$  to denote, respectively, the classical projective dimension, injective dimension and flat dimension of  $M$ . By  $\text{gldim}(R)$  and  $\text{wdim}(R)$  we denote, respectively, the classical global dimension and weak dimension of  $R$ .

Recall that the Gorenstein homological theory starts in the sixties with Auslander and Bridger [1, 2] over commutative Noetherian rings and developed, several decades later, by Enochs, Jenda, Christensen, Holm, Yassemi and others (see [3, 4, 5, 9, 10, 11, 14]).

Recently in [4], the authors started the study of global Gorenstein dimensions of rings, which are called, for a commutative ring  $R$ , Gorenstein global projective, injective, and weak dimensions of  $R$ , denoted by  $GPD(R)$ ,  $GID(R)$ , and  $G.wdim(R)$ , respectively, and, respectively, defined as follows:

- 1)  $GPD(R) = \sup\{Gpd_R(M) \mid M \text{ } R\text{-module}\}$
- 2)  $GID(R) = \sup\{Gid_R(M) \mid M \text{ } R\text{-module}\}$
- 3)  $G.wdim(R) = \sup\{Gfd_R(M) \mid M \text{ } R\text{-module}\}$

They proved that, for any ring  $R$ ,  $G.wdim(R) \leq GID(R) = GPD(R)$  ([4, Theorems 1.1 and Corollary 1.2(1)]). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of  $GPD(R)$  and  $GID(R)$  is called Gorenstein global dimension of  $R$ , and denoted by  $G.gldim(R)$ . They also proved that the Gorenstein global and weak dimensions are refinement of the classical global and weak dimensions of rings. That is :  $G.gldim(R) \leq \text{gldim}(R)$  and  $G.wdim(R) \leq \text{wdim}(R)$  with equality if  $\text{wdim}(R)$  is finite ([4, Corollary 1.2(2 and 3)]).

Let  $R$  be a ring and  $E$  an  $R$ -module. The trivial ring extension of  $R$  by  $E$  is the ring  $R := A \ltimes E$  whose underlying group is  $A \times E$  with multiplication given by

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$(r, e)(r', e') = (rr', re' + r'e)$  ([13, 15, 17, 18]). Over  $R \ltimes E$ , the module  $0 \times E$  is an ideal. Moreover, the diagonal embedding  $\varphi : R \rightarrow R \ltimes E$ , defined by  $\varphi(r) = (r, 0)$ , is an injective ring homomorphism. Hence we have the following short exact sequence of  $R$ -modules:

$$(*) \quad 0 \rightarrow R \xrightarrow{\varphi} R \ltimes E \xrightarrow{\psi} E \rightarrow 0$$

where  $\psi((r, e)) = e$ , for every  $(r, e) \in R \ltimes E$ . Notice that this sequence splits. We also have the short exact sequence of  $R \ltimes E$ -modules:

$$(**) \quad 0 \longrightarrow 0 \times E \xrightarrow{i} R \ltimes E \xrightarrow{\varepsilon} R \longrightarrow 0$$

where  $i$  is the injection and  $\varepsilon(r, e) = r$ . Note that  $R$  is an  $R \ltimes E$ -module via the map ring  $\varepsilon$  (explicitly for all  $r, r' \in R$  and  $e \in E$ ,  $(r, e).r' = \varepsilon(r, e)r' = rr'$ ). Contrarily to  $(*)$  this sequence never splits as shown by the following result:

**Proposition 1.1.** *Let  $R$  be a ring and  $E \neq 0$  an  $R$ -module. Then,  $R$  is never projective as an  $R \ltimes E$ -module.*

*Proof.* Consider the short exact sequence of  $R \ltimes E$ -modules:

$$(**) \quad 0 \longrightarrow 0 \times E \xrightarrow{i} R \ltimes E \xrightarrow{\varepsilon} R \longrightarrow 0$$

where  $i$  is the injection and  $\varepsilon(r, e) = r$ . It is clear that  $R$  is projective if, and only if,  $(**)$  splits. Hence, there is an  $R \ltimes E$ -morphism  $\pi : R \rightarrow R \ltimes E$  such that  $\varepsilon \circ \pi = \text{id}(R)$ . Set  $\pi(1) = (r, e_0)$ . Thus,  $1 = \varepsilon \circ \pi(1) = \varepsilon(r, e_0) = r$ . Hence, for an arbitrary  $r \in R$  and any  $e \in E$ , we have  $\pi(r) = \pi((r, e).1) = (r, e)(1, e_0) = (r, re_0 + e)$ . But that is impossible since  $\pi$  must be well defined and  $E \neq 0$ .  $\square$

More general, If  $E$  is a flat  $R$ -module, from [13, Corollary 4.7], we conclude:

**Lemma 1.2.** *Let  $R$  be a ring and  $E$  a flat  $R$ -module. Then,  $fd_{R \ltimes E}(R) \leq n$  if, and only if,  $E^n := \underbrace{E \otimes E \otimes \dots \otimes E}_n = 0$ .*

To give examples of Lemma 1.2, we have to think about rings which contain an idempotent element (i.e;  $a \in R$  such that  $a^n = 0$  for a positive integer  $n$ ).

The homological behavior and structure of the  $R \ltimes E$ -module  $R$  has an importance counterpart in the determination of the Gorenstein and classical dimensions of the ring  $R \ltimes E$ . Recall that (see [4, Proposition 2.6])

$$R \text{ is quasi-Frobenius} \iff G.\text{gldim}(R) = 0$$

Adding the Noetherian condition to [13, Corollary 4.36] we obtain the next corollary:

**Corollary 1.3.** *Let  $R$  be a Noetherian ring and  $E$  a finitely generated  $R$ -module. Then,  $R \ltimes E$  is quasi-Frobenius if, and only if, the following conditions hold:*

- (1)  $E$  and  $\text{Ann}_R(E)$  are injective  $R$ -modules,
- (2) The naturel map  $R \rightarrow \text{Hom}_R(E, E)$  is an epimorphism, and
- (3)  $\text{Hom}_R(E, \text{Ann}_R(E)) = 0$ .

In [19], the authors study the Gorenstein dimension in trivial ring extensions. Namely they proved that for an  $R$ -module  $E$  with finite flat dimension such that  $G.\text{gldim}(R) < \infty$ , we have  $G.\text{gldim}(R) \leq G.\text{gldim}(R \ltimes E) + fd_R(E)$  ([19, Theorem 2.4]). Moreover, in [19] we find examples of trivial ring extensions of rings with

infinite Gorenstein global dimension (see [19, Theorems 3.2 and 3.4]).

In this paper we need the condition  $\text{Tor}_{R \times E}^n(R, M) = 0$  for any positive integer  $n$  and all  $R \times E$ -module with finite projective dimensions. To give an example of this situation we take  $E = xR$  where  $x$  is a nonzero divisor. We have the short exact sequences of  $R \times xR$ -modules:

$$(1) \quad 0 \longrightarrow 0 \times xR \xrightarrow{\iota} R \times xR \xrightarrow{\psi} R \longrightarrow 0$$

$$(2) \quad 0 \longrightarrow R \xrightarrow{\mu} R \times xR \xrightarrow{\nu} 0 \times xR \longrightarrow 0$$

where  $\iota$  is the injection,  $\psi((r, xr') = r$ ,  $\mu(r) = (0, xr)$  and  $\nu(r, xr') = (0, xr)$ . Then, from (1) and (2), for every  $R \times E$ -module with finite projective dimension we have  $\text{Tor}_{R \times xR}^i(R, M) = 0$ .

*Discussion 1.4 (Modules over  $R \times E$ ).* Via the ring map  $\varepsilon : R \times E \rightarrow R$  defined by  $\varepsilon(r, e) = r$ , we can give every  $R \times E$ -module  $M$  a structure of  $R$ -module (by setting  $r.m := (r, 0)m$  for every  $r \in R$ ). Moreover, we can consider the  $R$ -morphism (which depend only to the modulation of  $M$  over  $R \times E$ );  $\rho : E \otimes M \rightarrow M$  defined by  $\rho(e \otimes m) = (0, e)m$  (see that  $\rho$  is well defined by the universal propriety of tensor product). This  $R$ -morphism satisfying the condition

$$(\mathcal{H}) : \quad \rho(e \otimes \rho(e' \otimes m)) = 0 \quad \text{for } e, e' \in E \text{ and } m \in M$$

Conversely, given a pair  $(M, \rho)$  where  $M$  is an  $R$ -module and  $\rho : E \otimes_R M \rightarrow M$  is an  $R$ -morphism which satisfied the condition  $(\mathcal{H})$ . We can give  $M$  an  $R \times E$ -module structure via  $\rho$ . Namely for  $e \in E$ ,  $r \in R$  and  $m \in M$

$$(r, e).m := rm + \rho(e \otimes m)$$

(to see that the condition  $(\mathcal{H})$  guaranties the modulation we have just to try to prove that  $(r, e)[(r', e').m] = [(r, e)(r', e')].m$ ). These two constructions are inverse of each other. Hence, we can identified an  $R \times E$ -module  $M$  to a pair  $(M, \rho)$  where  $\rho$  satisfies the condition  $(\mathcal{H})$ .

A revealing example is to examine how  $R \times E$  is identifying to a pair  $(R \times, \rho)$ . So, as above we define  $\rho$  as  $\rho(e \otimes (r, e')) = (0, e)(r, e') = (0, re)$ .

Recall some well-know results.

**Proposition 1.5.** *Let  $R$  be a ring and  $E$  be an  $R$ -module. Then, for any  $R$ -module  $M$  we have:*

- (1)  $pd_R(M) \leq pd_{R \times E}(M)$ ,
- (2)  $id_R(M) \leq id_{R \times E}(M)$ , and
- (3)  $fd_R(M) \leq fd_{R \times E}(M)$ .

*Proof.* (1) and (2) are the particular cases of [13, Lemmas 4.1 and 4.2].

(3) If  $fd_{R \times E}(M) < \infty$ , then by [20, Lemma 3.51 and Theorem 3.52], we have  $id_{R \times E}(\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = fd_{R \times E}(M)$ . But the  $R \times E$ -modulation over  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is defined by, for every  $(r, e) \in R \times E$  and  $f \in \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ ,  $(r, e).f : M \rightarrow \mathbb{Q}/\mathbb{Z}$  such that for any  $m \in M$ ,

$$((r, e).f)(m) = f((r, e).m) = f(rm) = rf(m)$$

Thus,  $(r, e).f = rf$ . Hence, by (2) above,  $id_R(Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq id_{R \ltimes E}(Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$ . Consequently, we have:

$$fd_R(M) = id_R(Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq id_{R \ltimes E}(Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) = fd_{R \ltimes E}(M).$$

□

## 2. MAIN RESULTS

The aim of this paper is to give a Gorenstein version of Proposition 1.5.

**Theorem 2.1.** *Let  $R$  be a ring and  $E$  an  $R$ -module such that  $pd_{R \ltimes E}(R) < \infty$ . Then, for any  $R$ -module  $M$  we have  $Gpd_R(M) \leq Gpd_{R \ltimes E}(M)$ . Consequently,  $G.gldim(R) \leq G.gldim(R \ltimes E)$ .*

To prove this Theorem we involve several Lemmas.

**Lemma 2.2.** *Let  $R$  be a ring and  $E$  an  $R$ -module such that  $pd_{R \ltimes E}(R) < \infty$ . If  $M$  is a Gorenstein projective  $R \ltimes E$ -module then  $M \otimes_{R \ltimes E} R$  is a Gorenstein projective  $R$ -module. Moreover, if  $Tor_{R \ltimes E}^i(M, R) = 0$  for all  $i > 0$  then  $Gpd_R(M \otimes_{R \ltimes E} R) \leq Gpd_{R \ltimes E}(M)$ .*

*Proof.* Note in first that for every Gorenstein projective  $R \ltimes E$ -module, we have  $Tor_{R \ltimes E}(M, R) = 0$ . Indeed, we can pick an exact sequence of  $R \ltimes E$ -modules:

$$0 \rightarrow M \rightarrow P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_n \rightarrow G \rightarrow 0$$

where all  $P_i$  are projective and for any integer  $n > 0$  (in particular for  $n := pd_{R \ltimes E}(R)$ ). Hence,  $Tor_{R \ltimes E}(M, R) = Tor_{R \ltimes E}^{n+1}(G, R) = 0$ . Recall also that a Gorenstein projective module is an image of a morphism in a complete projective resolution.

Let  $M$  be an arbitrary  $R \ltimes E$ -module and consider a complete projective resolution of  $R \ltimes E$ -modules:

$$\mathbf{P} : \dots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \dots$$

such that  $M = Im(P_0 \rightarrow P^0)$  ([14, Definition 2.1]). By the reason above, the operator  $-\otimes_{R \ltimes E} R$  leaves  $\mathbf{P}$  exact. Then, we obtain an exact sequence of  $R$ -modules:

$$\mathbf{P} \otimes_{R \ltimes E} R : \dots \rightarrow P_1 \otimes_{R \ltimes E} R \rightarrow P_0 \otimes_{R \ltimes E} R \rightarrow P^0 \otimes_{R \ltimes E} R \rightarrow P^1 \otimes_{R \ltimes E} R \rightarrow \dots$$

On the other hand, for each projective  $R$ -module  $Q$  we have  $pd_{R \ltimes E}(Q) \leq pd_{R \ltimes E}(R) < \infty$ . Thus,  $Hom_R(\mathbf{P} \otimes_{R \ltimes E} R, Q) \cong Hom_{R \ltimes E}(\mathbf{P}, Q)$  is exact ([14, Proposition 2.3]). Consequently,  $\mathbf{P} \otimes_{R \ltimes E} R$  is a complete projective resolution of  $R$ -modules. So,  $M \otimes_{R \ltimes E} R \cong Im((P_0 \rightarrow P^0) \otimes 1_R)$  is a Gorenstein projective  $R$ -module, as desired.

Now let  $M$  be an  $R \ltimes E$ -module with finite Gorenstein projective dimension equal to  $n$  such that  $Tor_{R \ltimes E}^i(M, R) = 0$  for all  $i > 0$ . The desired result follows by applying the functor  $-\otimes_{R \ltimes E} R$  to an  $n$ -step Gorenstein projective resolution of  $M$  over  $R \ltimes E$ . □

**Lemma 2.3.** *Let  $0 \rightarrow N \rightarrow N' \rightarrow N'' \rightarrow 0$  be an exact sequence of  $R$ -modules. Then,  $Gpd_R(N'') \leq \max\{Gpd_R(N'), Gpd_R(N) + 1\}$  with equality if  $Gpd_R(N') \neq Gpd_R(N)$ .*

*Proof.* Using [14, Theorems 2.20 and 2.24] the argument is analogous to the one of [7, Corollary 2, p. 135]. □

**Lemma 2.4.** *Let  $R$  be a ring and  $E$  an  $R$ -module such that  $pd_{R \ltimes E}(R) < \infty$ . Let  $B$  and  $D$  a couple of  $R$  modules and  $\rho : E \otimes_R (B \oplus D) \rightarrow B \oplus D$  which satisfies the condition  $(\mathcal{H})$  (see Discussion 1.4) and such that  $Im(\rho) \subseteq 0 \oplus D$ . With the identification of Discussion 1.4, we have  $Gpd_R(B) \leq Gpd_{R \ltimes E}((B \oplus D, \rho))$ .*

*Proof.* Recall that the  $R \ltimes E$ -modulation over  $(B \oplus D, \rho)$  is given by setting:

$$(r, e).(b, d) := r(b, d) + \rho(e \otimes (b, d))$$

(see Discussion 1.4).

In first, we assume that  $(B \oplus D, \rho)$  is a Gorenstein projective  $R \ltimes E$ -module and we claim that  $B$  is a Gorenstein projective  $R$ -module. Seeing that  $Im(\rho) = (0 \times I)(B \oplus D)$  and since  $R \cong R \ltimes E / (0 \times E)$ , it is clear that  $(B \oplus D)/Im(\rho) \cong (B \oplus D, \rho) \otimes_{R \ltimes E} R$  is a Gorenstein projective  $R$ -module (by Lemma 2.2). Now, consider the  $R$ -morphisms:  $(B \oplus D)/Im(\rho) \xrightarrow{\delta} B$  and  $B \xrightarrow{\delta'} (B \oplus D)/Im(\rho)$  defined by  $\delta(\overline{(b, d)}) = b$  and  $\delta'(b) = \overline{(b, 0)}$ . We can see easily that  $\delta$  is well defined. Indeed, if  $\overline{(b, d)} = \overline{(b', d')}$  then  $(b - b', d - d') \in Im(\rho)$  and so,  $b - b' = 0$  (since  $Im(\rho) \subseteq 0 \oplus D$ ). Also, we can check that  $\delta \circ \delta' = id(B)$ . Then,  $B$  is a direct summand of  $(B \oplus D)/Im(\rho)$ . Hence,  $B$  is a Gorenstein projective  $R$ -module (by [14, Theorem 2.5]). Therefore, we assume  $0 < n := pd_{R \ltimes E}((B \oplus D, \rho))$  and we proceed by induction on  $n$ . Inspecting the proof of [13, Lemma 4.1] we can construct a short exact sequence of  $R \ltimes E$ -modules with the form

$$0 \longrightarrow (K \oplus L, \phi) \longrightarrow Q \longrightarrow (B \oplus D, \rho) \longrightarrow 0$$

where  $Q$  is projective and  $Im(\phi) \subseteq 0 \oplus L (= L)$ . Hence, by the hypothesis induction and Lemma 2.3, we conclude that:

$$Gpd_{R \ltimes I}(B \oplus D, \rho) = 1 + Gpd_{R \ltimes I}(K \oplus L, \phi) \geq 1 + Gpd_R(K) \geq Gpd_R(B)$$

□

*Proof of Theorem 2.1.* Recall that the modulation of  $R \ltimes I$  over the  $R$ -module  $M$  is defined via the ring map  $R \ltimes I \rightarrow R$  defined by  $(r, r + i) \mapsto r$ . Explicitly, we have for all  $m \in M$ ,  $(r, r + i).m = rm$ . So, we can identify this  $R \ltimes I$ -module with the  $R \ltimes I$ -module  $(M, \rho)$  with  $\rho : I \otimes M \rightarrow M$  is the zero  $R$ -morphism. Thus, by Lemma 2.4,  $pd_R(M) \leq pd_{R \ltimes I}(M)$ , as desired. □

*Remark 2.5.* Notice that the hypothesis of Theorem 2.1 is sufficient but not necessary. A simple example to see that is by considering the ring  $R \ltimes R$  where  $R$  is coherent. Using [16, Theorem 1.4.5] we can prove that  $fd_{R \ltimes R}(R) = \infty$ . But,  $G.gldim(R \ltimes R) = G.gldim(R)$  ([6, Proposition 2.5]). In [13, Proposition 3.11 and Corollary 5.5], the authors give an other example of our remark. Namely, if  $E$  is a finitely generated projective module over a Noetherian ring  $R$  then,

$$R \ltimes E \text{ is } n\text{-Gorenstein} \implies R \text{ is } n\text{-Gorenstein}$$

Recall that  $R$  is called  $n$ -Gorenstein if it is Noetherian with  $id_R(R) \leq n$  and note that if  $R$  is a Noetherian ring then  $G.gldim(R) \leq n \Leftrightarrow R$  is  $n$ -Gorenstein (for  $\Rightarrow$  see [12] and for  $\Leftarrow$  use [14, Theorem 2.20]).

**Proposition 2.6.** *Let  $R$  be a ring and  $E$  an  $R$  module such that  $G.gldim(R \ltimes E) < \infty$ . Suppose that  $Tor_{R \ltimes E}^i(M, R) = 0$  for all  $i > n$  and every  $R \ltimes E$ -module  $M$  with finite projective dimension. Then,  $G.gldim(R \ltimes E) \leq G.gldim(R) + n$ .*

To prove this Proposition, we need the following Lemma.

**Lemma 2.7.** *Let  $R$  be a ring with finite Gorenstein projective dimension, then, for a positive integer  $n$ , the following statements are equivalent:*

- (1)  $G.\text{gldim}(R) \leq n$ ;
- (2)  $\text{pd}(I) \leq n$  for every injective module  $I$ .

*Proof.* Note that  $G.\text{gldim}(R) = \sup\{\text{Gid}(M) \mid M \text{ an } R\text{-module}\}$  (by [4, Theorem 1.1]). Thus, using [14, Theorem 2.22],  $G.\text{gldim}(R) \leq n \Leftrightarrow \text{Ext}^i(I, M) = 0$  for each  $i > n$  and for any injective module  $I$  and each module  $M$ . Thus,  $G.\text{gldim}(R) \leq n \Leftrightarrow \text{pd}(I) \leq n$  for each injective module  $M$ , as desired.  $\square$

*Proof of Proposition 2.6.* We may assume that  $m := G.\text{gldim}(R)$  and  $n$  are finite. Otherwise, the result is obvious. Let  $I$  be an arbitrary injective  $R \ltimes E$ -module. Since  $G.\text{gldim}(R \ltimes E) < \infty$ , we have  $\text{pd}_{R \ltimes E}(I) < \infty$  (by Lemma 2.7). For such module pick an  $n$ -step projective resolution as follows:

$$0 \rightarrow K \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow I \rightarrow 0$$

Hence,  $\text{Tor}_{R \ltimes E}^i(K, R) = 0$  for all  $i > 0$ . Thus, using [13, Theorem 4.9],  $\text{pd}_R(K \otimes_{R \ltimes E} R) = \text{pd}_{R \ltimes E}(K) < \infty$ . Then,  $\text{pd}_{R \ltimes E}(K) = \text{pd}_R(K \otimes_{R \ltimes E} R) \leq m$  (by [4, Corollary 2.7]). Consequently,  $\text{pd}_{R \ltimes E}(I) \leq G.\text{gldim}(R) + n$ . Thus, from Lemma 2.7, we obtain the desired result.  $\square$

**Corollary 2.8.** *Let  $R$  be a ring and  $E$  a non-zero cyclic  $R$  module such that  $G.\text{gldim}(R \ltimes E) < \infty$ . Then,  $G.\text{gldim}(R \ltimes E) \leq G.\text{gldim}(R)$ .*

*Proof.* Inspecting the proof of [13, Theorem 2.28] we see that for a cyclic  $R$ -module  $E$  we have:  $\text{Tor}_{R \ltimes E}^i(M, R) = 0$  for all  $i > 0$  and each  $R \ltimes E$ -module with finite projective dimension  $M$ . Thus, the desired result follows directly from Proposition 2.6.  $\square$

Now we give our second main result in this paper.

**Theorem 2.9.** *Let  $R$  be a ring and  $E$  an  $R$ -module such that  $R \ltimes E$  is coherent and such that  $\text{fd}_{R \ltimes E}(R) < \infty$ . Then, for any  $R$ -module  $M$  we have  $\text{Gfd}_R(M) \leq \text{Gfd}_{R \ltimes E}(M)$ . Consequently,  $G.\text{wdim}(R) \leq G.\text{wdim}(R \ltimes E)$ .*

First we have to recall that in [16, Theorem 4.4.4], Glaz gives the necessary and sufficient condition under  $R$  and  $E$  to obtain the coherence of  $R \ltimes E$  and make sure that if  $R \ltimes E$  is coherent, so is  $R$ .

**Lemma 2.10.** *Let  $R$  be a ring and  $E$  an  $R$ -module such that  $\text{fd}_{R \ltimes E}(R) < \infty$ . If  $M$  is a Gorenstein flat  $R \ltimes E$ -module then  $M \otimes_{R \ltimes E} R$  is a Gorenstein flat  $R$ -module.*

*Proof.* Let  $M$  be a Gorenstein flat  $R \ltimes E$ -module and consider a complete flat resolution of  $R \ltimes E$ -modules:

$$\mathbf{F} : \dots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$$

such that  $M = \text{Im}(F_0 \rightarrow F^0)$  ([14, Definition 3.1]). By the same reason that in the proof of Lemma 2.2, the operator  $-\otimes_{R \ltimes E} R$  leaves  $\mathbf{F}$  exact since  $\text{fd}_{R \ltimes E}(R) < \infty$ . So, we obtain the exact flat resolution of  $R$ -modules:

$$\mathbf{F} \otimes_{R \ltimes E} R : \dots \rightarrow F_1 \otimes_{R \ltimes E} R \rightarrow F_0 \otimes_{R \ltimes E} R \rightarrow F^0 \otimes_{R \ltimes E} R \rightarrow F^1 \otimes_{R \ltimes E} R \rightarrow \dots$$

Now let  $I$  be an injective  $R$ -module,  $N$  an arbitrary  $R \ltimes E$ -module and set  $fd_{R \ltimes E}(R) = n$ . Pick an  $n$ -step projective resolution of  $N$  over  $R \ltimes E$  as follows:

$$0 \rightarrow N' \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow N \rightarrow 0$$

Clearly,  $Tor_{R \ltimes E}(N', R) = Tor_{R \ltimes E}^{n+1}(N, R) = 0$ . Thus, from [8, Proposition 4.1.3], we have  $Ext_{R \ltimes E}(N', I) \cong Ext_R(N' \otimes_{R \ltimes E} R, I) = 0$ . Therefore,  $Ext_{R \ltimes E}^{n+1}(N, I) = Ext_{R \ltimes E}(N', I) = 0$ . Consequently,  $id_{R \ltimes E}(I) \leq n < \infty$ . Then, the complex  $\mathbf{F} \otimes_{R \ltimes E} R \otimes_R I \cong \mathbf{F} \otimes_{R \ltimes E} I$  is exact (direct consequence of [14, Theorem 3.14]) and so  $\mathbf{F} \otimes_{R \ltimes E} R$  is a complete flat resolution of  $R$ -modules. Therefore,  $M \otimes_{R \ltimes E} R = Im(F_0 \otimes_{R \ltimes E} R \rightarrow F^0 \otimes_{R \ltimes E} R)$  is a Gorenstein flat module.  $\square$

Using [14, Proposition 3.11] and the injective version of Lemma 2.3 we get the following Lemma:

**Lemma 2.11.** *Let  $0 \rightarrow N \rightarrow N' \rightarrow N'' \rightarrow 0$  be an exact sequence of modules over a coherent ring  $R$ . Then:  $Gfd_R(N'') \leq \max\{Gfd_R(N'), Gfd_R(N) + 1\}$  with equality if  $Gfd_R(N') \neq Gfd_R(N)$ .*

*Proof of Theorem 2.9.* Recall that  $R$  is also coherent (by [16, Theorem 4.4.4]). Similarly that in the proof of Lemma 2.4; by replacing Lemma 2.2, [14, Theorem 2.5] and Lemma 2.3 by Lemma 2.10, [14, Proposition 3.13] and Lemma 2.11 respectively, we prove that: if  $B$  and  $D$  are a couple of  $R$  modules and  $\rho : E \otimes_R (B \oplus D) \rightarrow B \oplus D$  which satisfies the condition  $(\mathcal{H})$  (see Discussion 1.4) and such that  $Im(\rho) \subseteq 0 \oplus D$ , then  $Gfd_R(B) \leq Gfd_{R \ltimes E}((B \oplus D, \rho))$ . Consequently, as in the proof of Theorem 2.1, we deduce that for any  $R$ -module  $M$  we have:  $Gfd_R(M) \leq Gfd_{R \ltimes E}(M)$ , as desired.  $\square$

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NAJIB MAHDOU, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY OF FEZ, BOX 2202, UNIVERSITY S.M. BEN ABDELLAH FEZ, MOROCCO., MAHDOU@HOTMAIL.COM

MOHAMMED TAMEKKANTE, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY OF FEZ, BOX 2202, UNIVERSITY S.M. BEN ABDELLAH FEZ, MOROCCO., TAMEKKANTE@YAHOO.FR